Definition

Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be a bounded function.

The lower and upper integrals of f on I are

$$L(f) := \sup\{L(f; P) : P \in \mathcal{P}(I)\}$$

$$U(f) := \inf\{U(f; P) : P \in \mathcal{P}(I)\}$$

Here, $L(f; P) := \sum_{k=1}^{n} \inf\{f(x) : x \in [x_{k-1}, x_k]\}(x_k - x_{k-1})$

$U(f; P) := \sum_{k=1}^{n} \sup\{f(x) : x \in [x_{k-1}, x_k]\}(x_k - x_{k-1})$

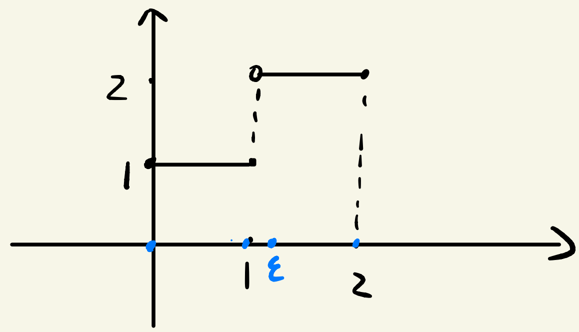
Theorem

- $L(f) \leq U(f)$
- $L(f) = U(f)$ if and only if f is Riemann integrable.

  In this case, $\int_a^b f\,dx = L(f) = U(f)$.

Example 1.

$$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 2, & 1 < x \le 2 \end{cases}$$



For any $\varepsilon > 0$, we define

$P_\varepsilon := (0, 1, 1+\varepsilon, 2)$.

Then $L(f; P_\varepsilon) = 1 \times 1 + \varepsilon \times 1 + (1-\varepsilon) \times 2 = 1 + \varepsilon + 2 - 2\varepsilon = 3 - \varepsilon$

$U(f; P_\varepsilon) = 1 \times 1 + \varepsilon \times 2 + (1-\varepsilon) \times 2 = 1 + 2\varepsilon + 2 - 2\varepsilon = 3$.

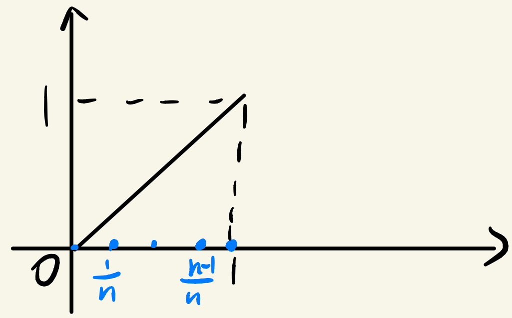
Then $L(f) \ge 3$ and $U(f) \le 3$.

Since $L(f) \le U(f)$, $L(f) = U(f) = 3$.

Therefore, $\int_0^2 f(x)\,dx = 3$.

Example 2.

$f(x) = x, \quad x \in [0,1]$.



For any $n \in \mathbb{N}$, we define

$$P_n := (0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1)$$

$$L(f; P_n) = \sum_{k=1}^{n} \frac{k-1}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{k=1}^{n} (k-1) = \frac{1}{n^2} \frac{(0+n-1)n}{2}$$

$$= \frac{n-1}{2n}$$

$$= \frac{1}{2}(1 - \frac{1}{n})$$

$$U(f; P_n) = \sum_{k=1}^{n} \frac{k}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{k=1}^{n} k = \frac{1}{n^2} \frac{(1+n)n}{2}$$

$$= \frac{n+1}{2n}$$

$$= \frac{1}{2}(1 + \frac{1}{n})$$

By def,

$$L(f) = \sup\{L(f;P) : P \in \mathcal{P}(I)\}$$

$$\geq \sup\{L(f; P_n) : n \in \mathbb{N}\} \qquad \{P_n : n \in \mathbb{N}\} \subset \mathcal{P}(I)$$

$$= \sup_{n \in \mathbb{N}} \{\frac{1}{2} - \frac{1}{n}\} = \frac{1}{2} \qquad \text{AP.}$$

Similarly, $U(f) \leq \frac{1}{2}$.

Since $\frac{1}{2} \leq L(f) \leq U(f) \leq \frac{1}{2}$, $L(f) = U(f) = \frac{1}{2}$.

Hence $\int_0^1 x\,dx = \frac{1}{2}$

Example 3:

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1] \\ 0, & x \in \mathbb{Q}^c \cap [0,1] \end{cases},$$

For any partition P of I, by the density of $\mathbb{Q}$,

$\sup\{f(x) : x \in [x_{k-1}, x_k]\} = 1$ for any k.

Thus $U(f;P) = \sum_{k=1}^{n} 1 \cdot (x_k - x_{k-1}) = \sum_{k=1}^{n} (x_k - x_{k-1}) = 1$.

Therefore, $U(f) = 1$

Similarly, by the density of $\mathbb{Q}^c$,

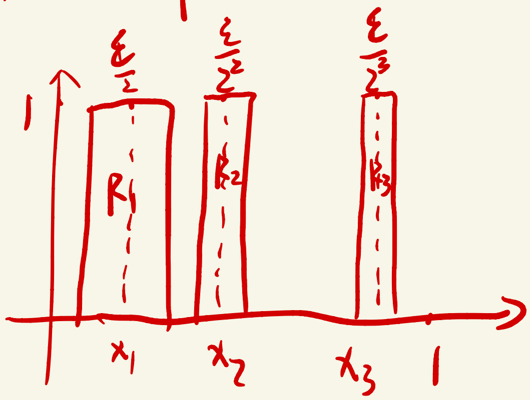
$\inf\{f(x) : x \in [x_{k-1}, x_k]\} = 0$ for any k.

Then $L(f;P) = \sum_{k=1}^{n} 0 (x_k - x_{k-1}) = 0$.

Therefore, $L(f) = 0$.

Since $L(f) = 0 \neq 1 = U(f)$, f is not integrable.

Rmk: The phenomenon is a little bit strange.



Since $\mathbb{Q}$ is countable, we can enumerate

$\mathbb{Q} = \{x_1, x_2, x_3, \dots\}$.

For any $\varepsilon > 0$, let R_n be the rectangles with side lengths 1 and $\frac{\varepsilon}{2^n}$.

Since $\{R_n\}_{n=1}^{\infty}$ covers the region under f.

Intuitively,
$$0 \le \text{Area}(f) \le \sum_{n=1}^{\infty} \text{Area}(R_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

Since ε can be arbitrarily small,

Area(f) should be 0.

So we expect $\int_0^1 f(x)dx = 0$.

But in the setting of Riemann integral, f is not integrable. That means there is some shortcoming in this setting.

In fact, we have 'Lebesgue integral' where f is integrable. (See MATH 4050 Real Analysis)